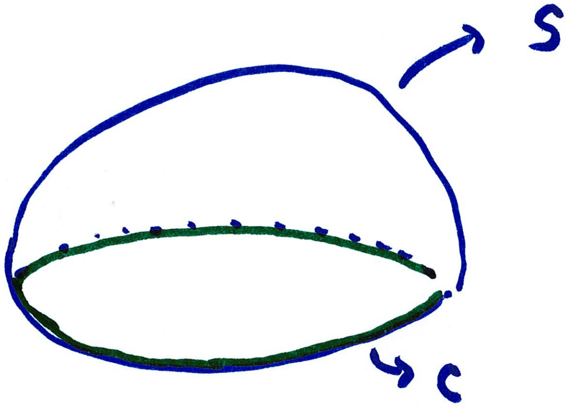
16.8 Stokes' Theorem

Stokes' Theorem relates a surface integral to a line integral in a vector field

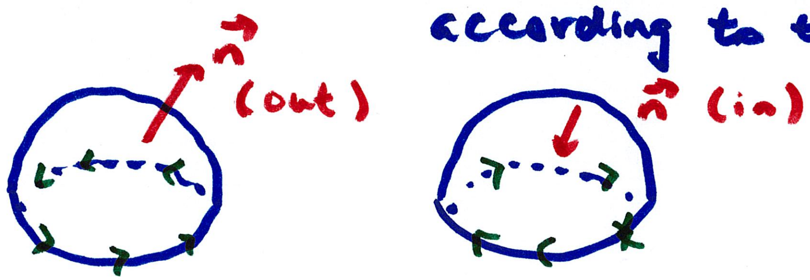


S: surface

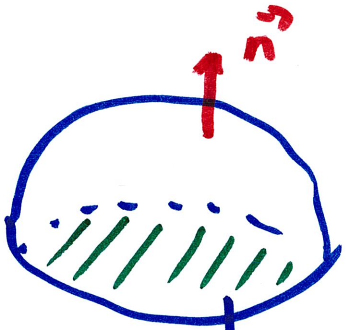
C: boundary of the surface

Stokes' Theorem: $$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$$

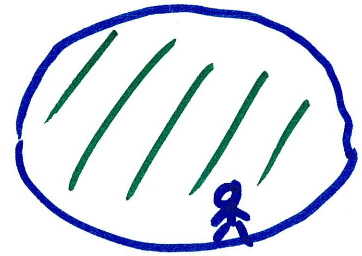
orientation: orientation of C is based on the orientation of S according to the right-hand rule



align thumb w/ $\vec{n}$, the direction fingers curl is the orientation of C



region enclosed by C



align head w/ $\vec{n}$

if walking around boundary, walk in direction such that region enclosed by C is on **LEFT**

Theorem is due to George Stokes (1819 - 1903)

and

William Thomson (1824 - 1907)

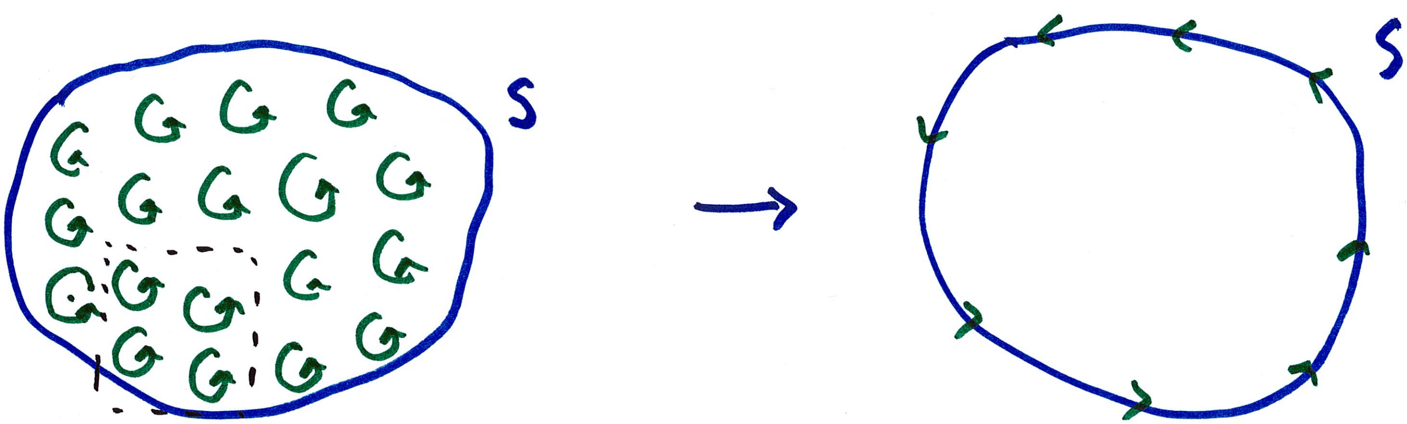
(aka 1st Baron kelvin)

↳ temperature scale

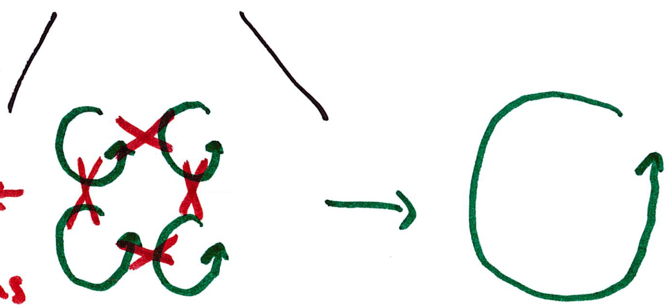
Why is Stokes' Theorem true?

why is $\underbrace{\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}} = \int_C \vec{F}\cdot d\vec{r}$?

accumulate
curl of vector field
over the surface



just the vector
field on the
boundary C



opposite
directions
(cancel)

vortices cancel
where they meet,
so behave like one big one

sometimes, $\int_C \vec{F} \cdot d\vec{r}$ is written as $\int_{\underbrace{\partial S}_{\text{boundary of } S}} \vec{F} \cdot d\vec{r}$

relation of Stokes' Theorem to $\underline{\text{Green's Theorem}}$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F} = \langle P, Q \rangle$ (2D only)

$\text{curl } \vec{F} = \vec{\nabla} \times \langle P, Q \rangle = \langle 0, 0, Q_x - P_y \rangle$

$\cancel{\iint_D (Q_x - P_y}$

$\vec{n}$ in 2D

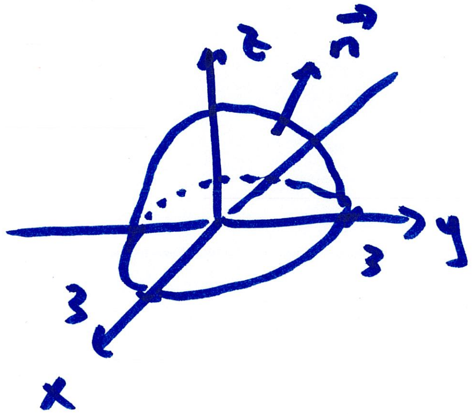
$$\iint_D \text{curl } \vec{F} \cdot \underbrace{\vec{k}\, dA}_{d\vec{S} \text{ for a 2D surface on } xy\text{-plane}}$$

So, Green's is a special case of Stokes'

<u>example</u> $\vec{F} = \langle y, -x, 0 \rangle$

$S: \ x^2+y^2+z^2 = 9 \ , \ z \geq 0$, $\underbrace{\text{oriented } \underline{\text{outward}}}$

$\vec{n}$ points <u>away</u> from origin
($\vec{k}$ component ≥ 0)



calculate $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$ as a surface integral

$\text{curl}\,\vec{F} = \cdots = \langle 0, 0, -2 \rangle = -2\vec{k}$

S: (spherical) $\vec{r}(u,v) = \langle 3\sin u \cos v, \ 3\sin u \sin v, \ 3\cos u \rangle$

$0 \leq u \leq \pi/2 \ , \quad 0 \leq v \leq 2\pi$

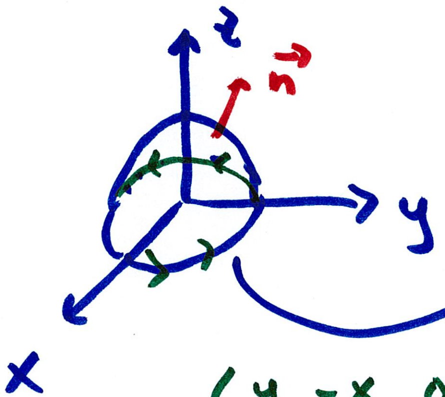
$\vec{r}_u \times \vec{r}_v = \cdots = \langle 9\sin^2 u \cos v, \ 9\sin^2 u \sin v, \ 9\sin u \cos u \rangle$

is this <u>outward</u> ? yes.

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} \langle 0,0,-2\rangle \cdot \langle \underbrace{\ \cdots\ }_{\vec{r}_u \times \vec{r}_v \text{ from last page}} \rangle\, du\, dv = \cdots = -18\pi$$

but it is MUCH easier as a line integral ~~ee~~ using Stokes' Theorem

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$$



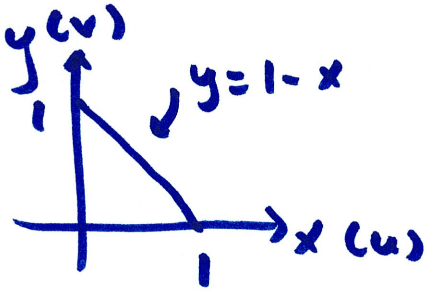
$$C: \vec{r}(t) = \langle 3\cos t,\ 3\sin t,\ 0\rangle$$
$$0 \le t \le 2\pi$$

$$d\vec{r} = \langle -3\sin t,\ 3\cos t,\ 0\rangle\, dt$$

$$\int_C \overbrace{\vec{F}}^{\langle y,-x,0\rangle}\cdot d\vec{r} = \int_0^{2\pi} \langle \underbrace{3\sin t}_{\substack{y \text{ component} \\ \text{from } \vec{r}(t)}},\ \underbrace{-3\cos t}_{\substack{-x \\ \text{comp.}}},\ 0\rangle \cdot \langle -3\sin t,\ 3\cos t,\ 0\rangle\, dt$$

$$= \int_0^{2\pi} (-9\sin^2 t - 9\cos^2 t)\, dt = \int_0^{2\pi} -9\, dt = -9\cdot 2\pi = -18\pi$$

$x+y+z=1 \rightarrow \vec{r}(u,v) = \langle u, v, 1-u-v \rangle$



$0 \le u \le 1, \quad 0 \le v \le 1-u$

$\vec{r}_u \times \vec{r}_v = \cdots = \langle 1, 1, 1 \rangle$ oriented out? yes.

$\text{curl } \vec{F} = \cdots = \langle -x, -2x, z-1 \rangle$

$= \langle -u, -2u, -u-v \rangle$

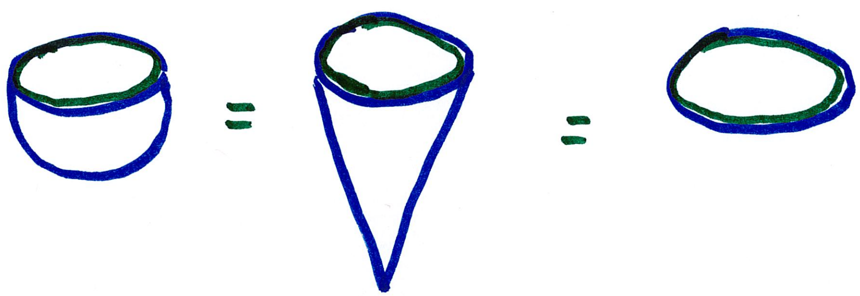
$$\iint_S \text{curl } \vec{F} \cdot d\vec{S}$$

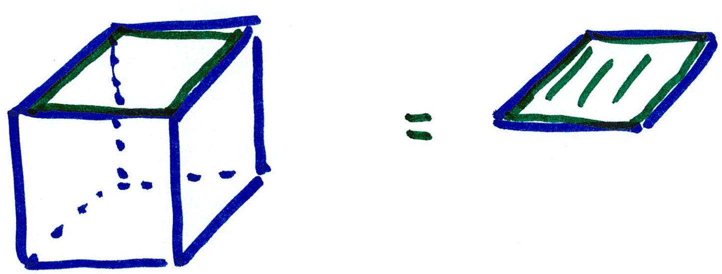
$$= \int_0^1 \int_0^{1-u} \langle -u, -2u, -u-v \rangle \cdot \langle 1, 1, 1 \rangle \, dv \, du$$

$$= \int_0^1 \int_0^{1-u} (-4u-v) \, dv \, du = \cdots = -\frac{5}{6}$$

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$$

→ if two surfaces have the same boundary, then their surface integrals are equal





cube w/o top

boundary: square at top

$\iint_S \text{curl}\,\vec{F}\cdot dS$ of cube w/o top

$= \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ over the top square

be careful w/ direction of $\vec{n}$!!